

[This question paper contains 4 printed pages.]

(16)

Your Roll No. 2023

Sr. No. of Question Paper : 1027

D

Unique Paper Code : 2352011101

Name of the Paper : DSC-1 : Algebra

Name of the Course : B.Sc. (H) Mathematics,
UGCF-2022

Semester : I

Duration : 3 Hours

Maximum Marks : 90



Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory and carry equal marks.
3. Attempt any two parts from each question.

1. (a) Prove that one root of $x^3 + px^2 + qx + r = 0$ is negative of another root if and only if $r = pq$.

(7.5)

- (b) Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, whose roots are in arithmetical progression.

(7.5)

P.T.O.

(c) Find all the integral roots of

$$x^7 + 2x^5 + 4x^4 - 8x^2 - 32 = 0. \quad (7.5)$$

2. (a) Find the polar representation of the complex number

$$z = \sin a + i(1 + \cos a), \quad a \in [0, 2\pi) \quad (7.5)$$

(b) Find $|z|$ and $\arg z$ for $z = \frac{(2\sqrt{3} + 2i)^8}{(1-i)^6} + \frac{(1+i)^6}{(2\sqrt{3} - 2i)^8}$

(7.5)

(c) Find the geometric image for the complex number z such that

$$|z + 1 + i| < 3 \text{ and } 0 < \arg z < \frac{\pi}{6} \quad (7.5)$$

3. (a) Let $U_n = \{\epsilon_0, \epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}\}$ be the set of n^{th} roots of unity, where $\epsilon_k = \cos(2k\pi/n) + i \sin(2k\pi/n)$, $k \in \{0, 1, 2, \dots, n-1\}$. Prove the following:

(i) Prove that $\epsilon_j \epsilon_k \in U_n$ for all $j, k \in \{0, 1, 2, \dots, n-1\}$

(ii) $\epsilon_j^{-1} \in U_n$ for all $j \in \{0, 1, 2, \dots, n-1\}$

(4+3.5)

- (b) Let a be an integer, show that there exists an integer k such that

$$a^2 = 3k \text{ or } a^2 = 3k + 1. \quad (7.5)$$

- (c) (i) Prove that $\gcd(n, n+1) = 1$ for every natural number n . Find integers x and y such that $n.x + (n+1).y = 1$.

- (ii) Let a , b and c be three natural numbers such that $\gcd(a, c) = 1$ and b divides c . Prove that $\gcd(a, b) = 1$. (4+3.5)

4. (a) Let $n > 1$ be a fixed natural number. Let a , b , c be three integers such that $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$. Prove that $a \equiv b \pmod{n}$. (7.5)

- (b) Solve the congruence, $7x \equiv 8 \pmod{11}$. (7.5)

- (c) Solve the following pair of congruences:

$$2x + 3y \equiv 1 \pmod{6}$$

$$x + 3y \equiv 5 \pmod{6} \quad (7.5)$$

5. (a) Let G be the set of all 2×2 real matrices with non-zero determinant. Show that G is a group under the operation of matrix multiplication. Further show that it is not an Abelian Group. (7.5)

- (b) Let G be a group such that for any x, y, z in the group, $xy = zx$ implies $y = z$ (called left-right cancellation property). Show that G is Abelian. Give an Example of a non-abelian group in which left-right cancellation property does not hold. (7.5)
- (c) Show that the set $G = \{1, 5, 7, 11\}$ is a group under multiplication modulo 12 with the help of the Cayley table. (7.5)
6. (a) Show that for any integer n , the set $H_n = \{n.x \mid x \in \mathbb{Z}\}$ is a subgroup of the group $\mathbb{Z}$ of integers under the operation of addition. Further show that $H_2 \cup H_3$ is not a subgroup of $\mathbb{Z}$. (5.5+2)
- (b) Let G be a group. Show that $|aba^{-1}| = |b|$ for all a and b in G ($|x|$ denotes the order of an element x in G). (7.5)
- (c) Show that the group $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ is cyclic under the operation of addition modulo n . How many generators $\mathbb{Z}_n$ have? Further, describe all the subgroups of $\mathbb{Z}_{40}$. (2+1+4.5)

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Your Roll No. 2023

Sr. No. of Question Paper : 1046

D

Unique Paper Code : 2352011102

Name of the Paper : DSC-2: Elementary Real Analysis

Name of the Course : B.Sc. (H) Mathematics (UGCF-2022)

Semester : I

Duration : 3 Hours

Maximum Marks : 90



Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **three** parts from each question.
3. **All** questions carry equal marks.

1. (a) If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for any $\epsilon > 0$, then show that $a = 0$.

(b) Find all values of x that satisfy $|x - 1| > |x + 1|$. Sketch the graph of this inequality.

(c) Find the supremum and infimum, if they exist, of the following sets:

(i) $\left\{ \cos \frac{n\pi}{2} : n \in \mathbb{N} \right\}$

(ii) $\left\{ \frac{x+2}{3} : x > 3 \right\}$

P.T.O.

(d) Show that $\sup \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\} = 1$.

2. (a) Let S be a non-empty subset of $\mathbb{R}$ that is bounded.

Prove that

$$\inf S = -\sup \{-s : s \in S\}$$

(b) State and prove the Archimedean Property of real numbers.

(c) If $S = \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}$, find $\inf S$ and $\sup S$.

(d) Define a convergent sequence. Show that the limit of a convergent sequence is unique.

3. (a) Using the definition of limit, show that

$$\lim_{n \rightarrow \infty} \frac{2n+3}{3n-7} = \frac{2}{3}$$

(b) Show that $\lim_{n \rightarrow \infty} \left(n^{1/n} \right) = 1$.

(c) State and prove the Sandwich Theorem for sequences.

(d) Show that every increasing sequence which is bounded above is convergent.

4. (a) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for all $n \geq 1$. Prove that $\langle x_n \rangle$ converges and find its limit.

(b) Prove that every Cauchy sequence is bounded.

(c) Show that the sequence $\langle x_n \rangle$ where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \text{ for all } n \in \mathbb{N},$$

does not converge.

(d) Find the limit superior and limit inferior of the following sequences:

$$(i) \quad x_n = (-2)^n \left(1 + \frac{1}{n} \right), \text{ for all } n \in \mathbb{N}$$

$$(ii) \quad x_n = (-1)^n \left(\frac{1}{n} \right), \text{ for all } n \in \mathbb{N}$$

5. (a) Show that the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ converges if and only if $|r| < 1$.

(b) Find the sum of the following series, if it converges,

$$\sum \frac{1}{(n+a)(n+a+1)}, : a > 0$$

(c) Find the rational number which is the sum of the series represented by the repeating decimal $0.\overline{15}$.

(d) Check the convergence of the following series:

$$(i) \quad \sum \frac{1}{\log n}, n \geq 2$$

$$(ii) \quad \sum \tan^{-1} \left(\frac{1}{n} \right)$$

6. (a) State the Ratio Test (limit form) for positive series. Using this test or otherwise, check the convergence of the following series:

(i) $\sum \left(\frac{n!}{n^n} \right)$

(ii) $\sum \left(\frac{n!}{e^n} \right)$

- (b) Check the convergence of the following series:

(i) $\sum_{n=2}^{\infty} \left(\frac{\log n}{n^2} \right)$

(ii) $\sum \left(\frac{n^{n^2}}{(n+1)^{n^2}} \right)$

- (c) Define absolute convergence of a series. Show that every absolutely convergent series is convergent. Is the converse true? Justify your answer.

- (d) Check the following series for absolute or conditional convergence :

(i) $\sum (-1)^{n+1} \left(\frac{n}{n^2 + 1} \right)$

(ii) $\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{n^2 + (-1)^n} \right)$

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Your Roll No. 2023

Sr. No. of Question Paper : 1267

D

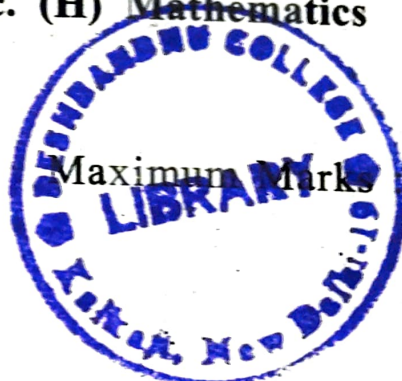
Unique Paper Code : 2352011103

Name of the Paper : DSC-3: Probability and Statistics

Name of the Course : B.Sc. (H) Mathematics

Semester : I

Duration : 3 Hours



Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory.
3. Attempt any **two** parts from each question.
4. All questions carry equal marks.
5. Use of non-programmable scientific calculators and statistical tables is permitted.

1. (a) A sample of 26 offshore oil workers took part in a simulated escape exercise, resulting in the following data on time (sec) to complete the escape. Construct a stem-and-leaf display and comment on any interesting features of the display.

P.T.O.

389	356	359	363	373	424	325	394	402	373
373	370	364	366	364	325	339	393	392	369
374	359	356	403	334	397				

- (b) The following data gives the sample of total nitrogen loads (kg N/day) from a particular Chesapeake Bay location. Calculate the median, upper fourth (third quartile) and lower fourth (first quartile).

9.69	13.16	17.09	18.12	23.70	24.07	24.29	26.43	30.75	31.54
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- (c) Data was collected for the sale of homes for homes in a particular city. The following data gives the sale amounts of homes (in 1000's of \$) that were sold in the previous month.

590	815	575	608	350	1285	408	540	555	679
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- (i) Calculate the sample variance and standard deviation.
- (ii) The evaluator realized that he had by mistake forgotten to multiply each observation by 5 while collecting data. What would be the resulting values of the sample variance and standard deviation after the correction? Answer without reperforming the calculations.

2. (a) A computer consulting firm presently has bids out on three projects. Let $A_k = \{\text{awarded project } k\}$, for $k = 1, 2, 3$, and suppose that $P(A_1) = .22$, $P(A_2) = .25$, $P(A_3) = .28$, $P(A_1 \cap A_2) = .11$, $P(A_1 \cap A_3) = .05$, $P(A_2 \cap A_3) = .07$, $P(A_1 \cap A_2 \cap A_3) = .01$. Compute the $P(A'_1 \cap A'_2)$ and $P(A_1 \cup A_2 \cup A_3)$.
- (b) State Baye's Theorem. An individual has three different email accounts. 70%, of her messages come into account A_1 , whereas 20% come into account A_2 and the remaining 10% into account A_3 . Of the messages into account A_1 , only 1% are spam, whereas the corresponding percentages for accounts A_2 and A_3 are 2% and 5%, respectively. A randomly selected mail is found to be a spam. What is the probability that it came in account A_2 ?
- (c) Components of a certain type are shipped to a supplier in batches of ten. Suppose that 50% of all such batches contain no defective components, 30% contain one defective component, and 20% contain two defective components. Two components from a batch are randomly selected and tested. What are the probabilities associated with 0, 1, and 2 defective components being in the batch under the condition that one of the two tested components is defective?

3. (a) Starting at a fixed time, the gender of each new born child is observed at a certain hospital until a boy (B) is born. Let $p = P(\text{boy is born}) = P(B)$ and assume that the successive births are independent. Let the random variable

X = the number of births upto and including that of the first boy

- (i) Find the probability mass function (pmf) of X .
- (ii) Determine the cumulative mass function (cmf) of X .

- (b) Let the random variable be defined as :

$X = 1$ if a randomly selected vehicle passes an emission test and $X = 0$ otherwise

Assume that the probability mass function (pmf) of this variable is: $p(1) = p$ and $p(0) = 1 - p$.

- (i) Name the random variable.
- (ii) Compute $E(X^2)$
- (iii) Show that $V(X) = p(1 - p)$

- (c) For any random variable X , let $E(X) = 5$ and $E[X(X - 1)] = 27.5$. Compute :

- (i) $E(X^2)$

(ii) $V(X)$

(iii) $V(2X + 3)$

4. (a) The error involved in making a certain measurement is a continuous random variable with probability density function as follows

$$f(x) = \begin{cases} .09375(4 - x^2) & -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (i) Obtain the cumulative density function $F(x)$ of X
- (ii) Compute $P(-1 < X < 1)$
- (iii) Compute $E[X]$
- (iv) Compute $V[X]$
- (b) Suppose that 25% of all students at a large public University receive financial aid. Let X be the number of students in a random sample of size 50 who receive financial aid. Using normal approximations find the approximate probabilities that
- (i) at most 10 students receive aid
- (ii) between 5 and 15 (inclusive) of the selected students receive aid.

(c) Define exponential distribution. Find the mean and standard deviation of an exponentially distributed random variable X . Are they equal?

5. (a) If a publisher of non technical books takes great pains to ensure that its books are free of typographical errors, so that the probability of any given page containing at least one such error is 0.005 and errors are independent from page to page, what is the probability that one of its 400 page novels will contain

(i) Exactly one page with errors?

(ii) At most three page with errors?

(b) An insurance company offers its policyholders a number of different premium payment options. For a randomly selected policyholder, let the random variable be defined as :

X = the number of months between successive payments with the cumulative density function(cdf) as follows :

$$F(x) = \begin{cases} 0 & x < 1 \\ .30 & 1 \leq x < 3 \\ .40 & 3 \leq x < 4 \\ .45 & 4 \leq x < 6 \\ .60 & 6 \leq x < 12 \\ 1 & 12 \leq x \end{cases}$$

(i) Determine the probability mass function of X ?

(ii) Using just the cdf, compute $P(3 \leq X \leq 6)$.

(c) The weight distribution of parcels sent in a certain manner is normal with mean value of 12 lb and standard deviation 3.5 lb. The parcel service wishes to establish a weight value c beyond which there will be a surcharge. What value of c is such that 99% of all parcels are under the surcharge weight?

6. (a) The Turbine Oil Oxidation Test (TOST) and Rotating Bomb Oxidation Test (RBOT) are two different procedures for evaluating the oxidation stability of steam turbine oils. The following table gives these observations on x = TOST time (in hours) and y = RBOT time (in minutes) for 10 oil specimens

X	4200	3600	3750	3675	4050	2770	4870	4500	3450	2700
y	370	340	375	310	350	200	400	375	285	225

(i) Calculate the value of the sample correlation coefficient. Based on this value, how would you describe the nature of relationship between the two variables?

(ii) If RBOT is also measured in hours, what happens to the value of r ? Why?

(b) The following table gives the data on x = rainfall volume (m^3) and y = runoff volume (m^3) for a particular location.

x	5	12	14	17	23	30	40	47	55	67
y	4	10	13	15	15	25	27	46	38	46

(i) Determine the equation of the estimated regression line using the principle of least square.

(ii) Estimate the runoff volume when the rainfall volume is 70.

(c) The amount of a particular impurity in a batch of a certain chemical product is a random variable with mean 4.0 g and standard deviation 1.5 g. If $\bar{X}$ is the sample mean impurity for a random sample of 50 batches.

(i) Where is the sampling distribution of $\bar{X}$ centered?

(ii) What is the standard deviation of the $\bar{X}$ distribution?

(iii) What is the probability that the sample average amount of impurity $\bar{X}$ is between 3.5 and 3.8 g?